

Gauge Independent Reduction of a Solvable Model with Gribov-Like Ambiguity

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Abstract

We present a gauge independent Lagrangian method of abstracting the reduced space of a solvable model with Gribov-like ambiguity, recently proposed by Friedberg, Lee, Pang and Ren. The reduced space is found to agree with the explicit solutions obtained by these authors. Complications related to gauge fixing are analysed. The Gribov ambiguity manifests by a nonuniqueness in the canonical transformations mapping the hamiltonian in the afflicted gauge with that obtained gauge independently. The operator ordering problem in this gauge is investigated and a prescription is suggested so that the results coincide with the usual hamiltonian formalism using the Schrödinger representation. Finally, a Dirac analysis of the model is elaborated. In this treatment it is shown how the existence of a nontrivial canonical set in the ambiguity-ridden gauge yields the connection with the previous hamiltonian formalism.

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I. Introduction

In a recent paper Friedberg, Lee, Pang and Ren [1] have proposed a solvable model which exhibits a Gribov-like ambiguity [2] that is known to exist in the Coulomb gauge of quantum chromodynamics. From the explicit solutions of the model, these authors have also shown that it is necessary to include all gauge-equivalent copies rather than adopting Gribov's suggestion of accounting for only those configurations having a positive Fadeev-Popov determinant. The explicit results given in [1] were obtained in the Hamiltonian approach using the Schrodinger representation and in the path integral approach using Feynman rules. Subsequently, a BRST analysis of this model was also carried out by Fujikawa [3]. Since many facets of the Gribov problem are clouded by the complications of a nonabelian gauge theory like quantum chromodynamics, the explicit computations possible in the solvable model [1] provide an insight that cannot be otherwise gained.

It is obvious that the model [1] under consideration is a gauge theory otherwise the issue of Gribov ambiguity does not arise. The conventional approach to isolate the true (physical) degrees of freedom from the unwanted (unphysical) ones, which is characteristic of a gauge theory, is to fix a gauge. In the hamiltonian formulation one usually starts from the time axial gauge where the Cartesian basis in the Schrödinger representation is defined. Transition to other gauges is achieved by coordinate transformations from the results in the time axial gauge [4]. This was the approach adopted in [1]. It was also shown that the mapping from the time axial gauge to a particular gauge was not one-to-one which was a manifestation of a Gribov-like ambiguity in that gauge.

There is, however, an alternative way [5], [6] of obtaining the reduced (physical) hamiltonian without fixing any gauge. In the hamiltonian formulation this reduction is based on the Levi-Civita transformations [7]. The viability or admissibility of any gauge is then shown by demanding canonical equivalence with the gauge independent result. Nevertheless, a constructive prescription for carrying out this gauge independent reduction is still lacking. Recently we [8] have developed a purely Lagrangian approach of systematically reducing the degrees of freedom in a gauge independent manner. The physical hamiltonian is then obtained directly from this reduced Lagrangian.

Gauge fixing can also be implemented and its results are analysed by discussing the canonical equivalence with the gauge independent computations. A positive feature of this approach is that the gauge independent analysis clearly reveals the most *natural* choice for gauge.

In this paper, using our aforesaid methods [8], a gauge independent Lagrangian reduction of the model [1] will be presented in section III. The physical hamiltonian, which is obtained directly from this Lagrangian, is expressed in terms of the independent canonical pairs. Different gauge fixings will be considered and the hamiltonian in these gauges will be explicitly computed. It is shown that the canonical transformations mapping the hamiltonian in a particular gauge with the hamiltonian obtained gauge independently does not possess a unique inverse. This is the manifestation of the Gribov-like ambiguity. Incidentally this nonuniqueness can be exactly identified with the nonuniqueness present in the coordinate transformations considered in the usual hamiltonian approach [1]. Our analysis shows that all gauge copies must be treated equivalently which is compatible with the proposal made in [1]. Additionally, it is found that the hamiltonian in the gauge which suffers from the Gribov problem is plagued by ordering ambiguities. A definite ordering prescription is given such that the gauge fixed hamiltonian reproduces the corresponding expression obtained in [1] using the Schrödinger representation. As an alternative hamiltonian formalism we have analysed this model in section IV employing Dirac's [9] theory of constrained systems. The complications of the Gribov problem are now revealed by a nontrivial pair of canonical variables in the relevant gauge. By comparing this nontrivial set with the standard canonical pairs found in Gribov ambiguity free gauges, it is possible to reconstruct exactly the coordinate transformations [1] mapping the hamiltonians in the distinct gauges. This establishes the connection of our Dirac analysis with the hamiltonian formalism of [1]. To illustrate our ideas in a simpler, yet highly relevant, setting the gauge independent Lagrangian reduction of the Christ Lee model [10] has been presented in section II. Indeed many of the physical concepts and algebraic manipulations developed here will be useful for section III. Our concluding remarks are given in section V.

We now give a brief review of the method [8] of gauge independently reducing the degrees of freedom in a given Lagrangian. An application to electrodynamics is also included which serves to illuminate several distinguishing features, particularly its close connection with the models [10], [1]

considered in subsequent sections. From the theory of differential equations unsolvable with respect to the highest derivatives, it is possible to express the lagrange equations for second order systems with variables v by an equivalent set of independent equations [5],

$$\ddot{p} = \Theta(p, \dot{p}, q, \beta, \dot{\beta}, \ddot{\beta}) \quad (1)$$

$$\dot{q} = \Phi(p, \dot{p}, q, \beta, \dot{\beta}) \quad (2)$$

$$r = \Psi(p, q, \beta) \quad (3)$$

where $v = (p, q, r, \beta)$ and Θ, Φ, Ψ are some functions of the indicated arguments. Note that an overdot denotes a time derivative. In a nonsingular theory q, r, β are absent so that there is an unconstrained dynamics with $\ddot{p} = \Theta(p, \dot{p})$. For singular theories (2) and (3) represent the constraints. Now recall that the lagrange equations were derived by a variational principle on the assumption that all v, \dot{v} were free. Since the constraints impose certain restrictions on v, \dot{v} , it is essential that these keep the above set of equations unmodified, or internal consistency is lost and the starting Lagrangian is not valid. Consequently time derivatives of the constraints must vanish by virtue of this set of equations. This implies that the complete constraint sector is given by (2) and (3).

The idea is now to pass from the constrained $v = (p, q, r, \beta)$ to the unconstrained $v = p$ by removing q, r, β . The variable r is trivially eliminated in favour of p, q, β by using (3). In the physically interesting gauge systems the constraints are implemented by a lagrange multiplier whose time derivative, therefore, does not appear in the Lagrangian. This multiplier is identified with q which can thus be removed in favour of p, β by using (2). The Lagrangian in the reduced sector is now a function of $(v, \dot{v}; v = p, \beta)$. By evaluating the lagrange equations in this sector it is possible to identify β with the variable that does not occur in these equations (see (1) to (3)). With this identification the variable β , which reflects the degeneracy in the system, automatically drops out from the Lagrangian and its final unconstrained form is obtained. The physical hamiltonian is now found by performing the standard Legendre transformation. Note that the usage of any gauge fixing has been completely avoided to obtain the reduced space. This analysis also indicates the most natural choice of gauge as that which implies the vanishing of β . In that case the reduced space obtained by gauge fixing would be trivially equivalent to the gauge independent reduction. For an arbitrary

gauge, however, it becomes necessary to check the canonical equivalence between the gauge fixed and gauge independent results, otherwise the gauge is not admissible.

An instructive illustration [8] of this gauge independent Lagrangian reduction is provided by the classic example of spinor electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\partial} - m - e\cancel{A})\psi \quad (4)$$

The equations of motion are,

$$(i\cancel{\partial} - m - e\cancel{A})\psi = 0 \quad (5)$$

$$\partial^\alpha F_{\alpha\mu} - ej_\mu = 0 \quad (6)$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$ is the current. The $\mu = 0$ component of (6) is a constraint. It can also be checked that this constraint is conserved in time by virtue of the equations of motion. Furthermore there is a degeneracy in these equations which becomes obvious from current conservation. The multiplier A_0 (identified with q) can be eliminated in favour of the other variables by solving the constraint. Using this, (4) is expressed in terms of the reduced set of variables. The Lagrange equations in these variables are [8],

$$\partial^j F_{ji} + \partial_0^2[(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2})A_j] + \frac{\partial_i}{\partial^2}\partial_0 j_0 - j_i = 0 \quad (7)$$

It is obvious that the variable β , manifesting the degeneracy, is just the longitudinal (L)- component of A_i which has dropped out from (7). Consequently by choosing the orthogonal polarisation,

$$A_i = A_i^T + A_i^L \quad (8)$$

the Lagrangian gets further reduced,

$$\mathcal{L} = \frac{1}{2}(\dot{A}_i^T)^2 - \frac{1}{4}F_{ij}^2(A^T) + \frac{1}{2}j_0\frac{1}{\partial^2}j_0 + j_i A_i^T + \mathcal{L}_M \quad (9)$$

where, expectedly, A_i^L gets automatically removed and \mathcal{L}_M is the pure matter part. The unconstrained Lagrangian is expressed in terms of the transverse (T)- component of A_i , which is the physical (gauge invariant) variable. The reduced Hamiltonian [8] obtained from this Lagrangian coincides with that

found in the hamiltonian formalism [5] of abstracting the canonical set by a Levi-Civita [7] transformation and then evaluating the total hamiltonian on the constraint surface. It is now clear that the natural gauge choice would be the Coulomb gauge $\partial_i A_i = 0$, since this implies $A_i^L = 0$. Indeed the Lagrangian (9) obtained without gauge fixing is exactly the Coulomb gauge-fixed Lagrangian found in the literature [5]. The investigations in the following sections provide further illustration and elaboration of the gauge independent Lagrangian reduction.

II. The Christ Lee Model

A simple model put forward by Christ and Lee (CL)[10] has been useful for testing different approaches [6], [11]. Furthermore, since a straightforward generalisation of this model leads to that considered in [1], which will be discussed in the next section, it provides a convenient starting point for the analysis. Indeed, as stated before, many physical concepts and algebraic manipulations introduced here will also be exploited in the next section. We begin with a gauge independent Lagrangian reduction which is followed by a gauge fixed computation. The connection with the usual hamiltonian formalism using the Schrödinger representation is established.

The CL model considers the motion of a point particle in two dimensional space whose dynamics is governed by the Lagrangian,

$$L(x_i, \dot{x}_i, q) = \frac{1}{2}\dot{x}_i^2 - \epsilon_{ij}x_i\dot{x}_j q + \frac{1}{2}q^2 x_i^2 - V(\rho^2) \quad (10)$$

where $x_i = x_1, x_2$ are the rectilinear coordinates of the two dimensional vector $\vec{\rho}$ so that $x_i^2 = x_1^2 + x_2^2 = \rho^2$, and q is another coordinate whose time derivative is absent in L so that it plays a role analogous to the multiplier A_0 in QED. The antisymmetric tensor ϵ_{ij} is defined such that $\epsilon_{12} = 1$. The equations of motion obtained by varying x_i and q are given by,

$$X_i = \ddot{x}_i + 2\epsilon_{ij}q\dot{x}_j + \epsilon_{ij}\dot{q}x_j - q^2 x_i + \frac{\partial V}{\partial x_i} = 0 \quad (11)$$

$$Q = q\rho^2 - \epsilon_{ij}x_i\dot{x}_j = 0 \quad (12)$$

There is a degeneracy or arbitrariness in these equations since,

$$\epsilon_{ij}x_iX_j + \dot{Q} = 0 \quad (13)$$

This is related to the invariance of the Lagrangian under the following infinitesimal gauge transformations,

$$x'_i = x_i - \theta(t)\epsilon_{ij}x_j \quad (14)$$

$$q' = q + \dot{\theta}(t) \quad (15)$$

This model, therefore, exhibits features similar to a gauge theory like QED, except that the abelian group here comprises rotations instead of translations. Furthermore it is simple to identify the constraint, which is the generator of gauge transformations, as being given by (12). This equation does not involve accelerations so that q can be determined from the initial Cauchy data. Moreover if (12) is satisfied at one time, it remains valid at all times since $\dot{Q} = 0$ by virtue of (11) and (13). This brings out the exact analogy of q with A_0 , and that of (12) with the Gauss constraint of QED. Following our general strategy q is now eliminated from (10) by using (12) to yield a reduced Lagrangian,

$$L(x_i, \dot{x}_i) = \frac{(x_i\dot{x}_i)^2}{2x_j^2} - V(\rho^2) \quad (16)$$

This is in exact analogy with the removal of A_0 in (4) using the Gauss constraint. It is now easy to see that this L does not depend on x_1 and x_2 independently, but only on their combination $(x_1^2 + x_2^2)$. Introducing, therefore, the polar decomposition,

$$\begin{aligned} x_1 &= \rho \cos \phi \\ x_2 &= \rho \sin \phi \end{aligned} \quad (17)$$

it is clear that the redundant or cyclic variable that does not appear in L is ϕ whereas ρ is the physical variable. In this variable we obtain,

$$L(\rho, \dot{\rho}) = \frac{1}{2}\dot{\rho}^2 - V(\rho^2) \quad (18)$$

as the final *unconstrained* form of the Lagrangian. It just represents the motion of the particle in one dimension subjected to the potential $V(\rho^2)$.

The reduced hamiltonian is given by,

$$\begin{aligned} H(\rho, \pi_\rho) &= \pi_\rho \dot{\rho} - L(\rho, \dot{\rho}) \\ &= \frac{1}{2} \pi_\rho^2 + V(\rho^2) \end{aligned} \quad (19)$$

where $\pi_\rho = \dot{\rho}$ is the momentum conjugate to ρ .

This completes the gauge independent reduction of the CL model within a purely Lagrangian approach. We have found the physical hamiltonian expressed in terms of the canonical variables. At this juncture it is worthwhile to make certain comments. The process of eliminating the unphysical or redundant variables is similar in both QED and the CL model. The Lagrange multiplier A_0 or q is first removed by exploiting the constraint. Then the cyclic coordinate A_i^L or ϕ is identified which automatically drops out from the reduced Lagrangian without any gauge fixing. To complete the analogy, ρ and ϕ in the CL model play the roles of A_i^T and A_i^L , respectively, in QED. Nevertheless, inspite of these similarities, there is a basic difference in the manner in which the cyclic coordinate is revealed. In QED an orthogonal polarisation (8), which is effectively a way of expressing A_i in terms of shifted variables, is required whereas in the CL model a curvilinear transformation (17) has to be performed. This is related to the fact that the abelian group in QED is translational while it is rotational in the other case.

Let us next consider the issue of gauge fixing. Making the standard choice of gauge [6],[11],

$$x_2 = \lambda x_1 \quad (20)$$

where λ is a real parameter, we find the corresponding solution for the multiplier q from (12),

$$q = 0 \quad (21)$$

since $x_1 = x_2 = 0$ is a singular point. The reduced Lagrangian following from the simultaneous imposition of (20) and (21) in (10) is given by,

$$L = \frac{1}{2}(1 + \lambda^2)\dot{x}_1^2 - V((1 + \lambda^2)x_1^2) \quad (22)$$

The canonical momenta is,

$$\pi_1 = \frac{\partial L}{\partial \dot{x}_1} = (1 + \lambda^2)\dot{x}_1 \quad (23)$$

and the hamiltonian is found by a simple Legendre transform,

$$H = \pi_1 \dot{x}_1 - L = \frac{1}{2} \frac{\pi_1^2}{1 + \lambda^2} + V((1 + \lambda^2)x_1^2) \quad (24)$$

If we now make the following canonical transformation,

$$\begin{aligned} x_1 &= (\sqrt{1 + \lambda^2})^{-1} \rho \\ \pi_1 &= (\sqrt{1 + \lambda^2}) \pi_\rho \end{aligned} \quad (25)$$

where (ρ, π_ρ) constitutes the new canonical pair, it is seen that the hamiltonian (24) reduces to the expression found in (19). Furthermore the canonical transformation (25) is nonsingular and the inverse transformation can be trivially read-off. Thus the dynamics obtained from the gauge fixed approach is canonically equivalent to that found in the gauge independent method. This implies that the above choice (20) of gauge is allowed.

It is now instructive to physically unravel the meaning of the gauge independent and gauge fixed analysis, including their canonical equivalence. The first step in this direction is to introduce the momenta conjugate to x_i in (10),

$$\pi_i = \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i + \epsilon_{ij} x_j q \quad (26)$$

In terms of this momenta the constraint (12) becomes,

$$Q = \epsilon_{ij} \pi_i x_j = 0 \quad (27)$$

This implies the vanishing of the angular momentum so that the motion of the particle is confined along a straight line. As far as the physics is concerned the slope of this line is inconsequential. The gauge independent analysis elegantly reproduces this dynamics by systematically removing the angular variable ϕ , thereby yielding the reduced hamiltonian (19). Now choosing the gauge (20) merely fixes the slope of this line as $\tan \phi = \lambda = \frac{x_2}{x_1}$. Consequently the dynamics in this gauge is canonically equivalent to that obtained gauge independently. Indeed the magnitude of the vector $\vec{\rho}$ (17), expressed in terms of the rectilinear coordinates x_1 and x_2 in the above gauge, is given by,

$$\rho^2 = x_1^2 + x_2^2 = (1 + \lambda^2)x_1^2 \quad (28)$$

which is just the inverse canonical transformation defined in (25). This completes our analysis of the CL model. Note that in the entire discussion the

choice of a specific Cartesian basis has been avoided since it was not necessary to introduce the Schrodinger representation. Nevertheless, as shown below, it is simple to illustrate the connection with the conventional hamiltonian approach [4] of choosing a Cartesian basis in the ‘time-axial’ gauge and subsequently passing to other gauges by appropriate gauge transformations.

The time-axial gauge in the CL model corresponds to taking $q = 0$. In this gauge the hamiltonian reduces to,

$$H = \frac{1}{2}\pi_i^2 + V(\rho^2) \quad (29)$$

where, using (26), $\pi_i = \dot{x}_i$ with x_1, x_2 being the Cartesian coordinates. As usual, the Gauss constraint (27) does not emerge from the analysis but has to be imposed by hand on the physical states. This constraint can be solved by using the curvilinear transformation (17), with the consequence that the physical states become independent of ϕ . Correspondingly, the hamiltonian (29) in these variables has the form [4],

$$H = -\frac{1}{2\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + V(\rho^2) \quad (30)$$

This result is the analogue of (19) with the identification $\pi_\rho^2 \rightarrow -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$.

Let us next consider the transition from the time-axial ($q = 0$) gauge with cartesian coordinates x_1, x_2 to the gauge (20) with coordinates X_1, X_2 ($q \neq 0$) brought about by (14) and (15),

$$X_1 = x_1 \cos \theta - x_2 \sin \theta \quad (31)$$

$$\lambda X_1 = x_1 \sin \theta + x_2 \cos \theta \quad (32)$$

$$q = \dot{\theta} \quad (33)$$

Equations (31), (32) define the coordinate transformations from (x_1, x_2) to (X_1, θ) . Using the Cartesian representation in the former set and the above transformations it can be shown that the Gauss constraint (27) in the latter (X_1, θ) basis is proportional to $(\frac{\partial}{\partial\theta})$. Repeating the above steps for the hamiltonian (29) and dropping terms proportional to $(\frac{\partial}{\partial\theta})$, which enforces the Gauss constraint, finally yields,

$$H = -\frac{1}{2}(1 + \lambda^2)^{-1}\frac{1}{X_1}\frac{\partial}{\partial X_1}X_1\frac{\partial}{\partial X_1} + V((1 + \lambda^2)X_1^2) \quad (34)$$

which just reproduces (24) where, as before, the momenta π_1 conjugate to X_1 is identified by $\pi_1^2 \rightarrow -\frac{1}{X_1} \frac{\partial}{\partial X_1} X_1 \frac{\partial}{\partial X_1}$. This shows the equivalence between the reduced space hamiltonian obtained in the conventional formulation [4] of doing a coordinate transformation from the (Cartesian) $q = 0$ gauge result and that found directly by the Lagrangian approach. It is, however, important to stress that contrary to the hamiltonian approach using the coordinate transformations, the general method of Lagrangian reduction presented here does not require any gauge fixing. Furthermore the validity of any gauge fixed computation has to be established by revealing its canonical equivalence with the gauge independent result. For the specific choice (20) this was explicitly elaborated.

III. Extended Christ Lee Model

An extended version of the CL model proposed recently by Friedberg, Lee, Pang and Ren [1] provides an interesting example of a solvable model with a Gribov-like ambiguity. This model has been analysed by these authors in details using the path integral and hamiltonian formulations. In the latter approach a "time-axial" gauge, in which the Cartesian basis was defined, was taken as the starting point. Transition to other gauges was achieved by performing curvilinear coordinate transformations from the time-axial gauge, exactly as shown here in the previous section for the CL model. It was then demonstrated that the mapping from the time-axial gauge to the so-called " λ gauge" [1] was not one-to-one, thereby realising a situation analogous to the well known Gribov [2] ambiguity. Here we shall apply our ideas elaborated in the preceding sections to provide a gauge independent way of abstracting the reduced (physical) space of this model. The reduced space will then be derived by choosing a gauge. As announced earlier, the Gribov ambiguity gets exposed by a *non-uniqueness* in the canonical transformation mapping the reduced hamiltonian in the " λ gauge" with the gauge independent result. Connection of our results with the usual canonical hamiltonian formalism based on coordinate transformations from the time-axial gauge will be discussed.

The model under consideration is defined by a simple extension of (10)

brought about by the introduction of a third Cartesian coordinate z , such that the new Lagrangian reads,

$$L(x_i, \dot{x}_i, \dot{z}, q) = \frac{1}{2}\dot{x}_i^2 - g\epsilon_{ij}x_i\dot{x}_jq + g^2\frac{1}{2}q^2x_i^2 + \frac{1}{2}(\dot{z} - q)^2 - V(x_1^2 + x_2^2) \quad (35)$$

where a coupling parameter $g > 0$ has also been explicitly inserted. The equation of motion obtained by varying x_i is identical to (11), except that q gets replaced by gq . The other Lagrange's equations are given by,

$$Z = \ddot{z} - \dot{q} = 0 \quad (36)$$

$$\tilde{Q} = g^2qx_i^2 - g\epsilon_{ij}x_i\dot{x}_j - \dot{z} + q = 0 \quad (37)$$

where the second equation is the analogue of (12). It is easy to see that the Lagrange equations are degenerate because of the relation,

$$\epsilon_{ij}x_iX_j + Z + \dot{\tilde{Q}} = 0 \quad (38)$$

which resembles the relation (13) in the CL model. This degeneracy is connected to the invariance of (35) under the gauge transformations defined by (14), (15) (with θ replaced by $\frac{\theta}{g}$) and,

$$z' = z + \frac{\theta}{g} \quad (39)$$

Exactly as happened for the CL model, the variable q acts as a Lagrange multiplier. It is determined from the initial Cauchy data by the constraint (37). Furthermore this constraint is fixed in time, as expected, since $\dot{\tilde{Q}} = 0$ by virtue of the degeneracy condition (38) and the equations of motion (11) and (36). Thus, as before, we solve (37) to eliminate q from the Lagrangian (35). The solution is given by,

$$q = \frac{\dot{z} + g(\epsilon_{ij}x_i\dot{x}_j)}{1 + g^2(x_1^2 + x_2^2)} \quad (40)$$

At this point we use results from the previous section to simplify the algebra. Noticing that the original rotational invariance in the $x_1 - x_2$ plane of the CL model is still preserved we use the polar decomposition (17) in terms of which the solution (40) reduces to,

$$q = \frac{\dot{z} + g\rho^2\dot{\phi}}{1 + g^2\rho^2} \quad (41)$$

Using this result the reduced Lagrangian, expressed in terms of polar variables, obtained from (35) is,

$$L = \frac{1}{2}[\dot{\rho}^2 + (1 + g^2\rho^2)^{-1}\rho^2(\dot{\phi} - g\dot{z})^2] - V(\rho^2) \quad (42)$$

It is clear that L does not depend on ϕ and z independently, but only on their combination $(\phi - gz)$. Consequently it is trivial to identify the redundant or cyclic variable that does not appear in the Lagrangian. Introducing a new variable,

$$\Phi = \phi - gz \quad (43)$$

the final reduced Lagrangian L_r in terms of unconstrained variables is given by,

$$L_r = \frac{1}{2}[\dot{\rho}^2 + (1 + g^2\rho^2)^{-1}\rho^2(\dot{\Phi})^2] - V(\rho^2) \quad (44)$$

The canonically conjugate momenta are given by,

$$\pi_\rho = \dot{\rho} \quad (45)$$

$$\pi_\Phi = \frac{\rho^2}{1 + g^2\rho^2}\dot{\Phi} \quad (46)$$

The reduced hamiltonian follows by taking a standard Legendre transform,

$$\begin{aligned} H &= \pi_\rho\dot{\rho} + \pi_\Phi\dot{\Phi} - L_r \\ &= \frac{1}{2}\pi_\rho^2 + \frac{1}{2}(g^2 + \frac{1}{\rho^2})\pi_\Phi^2 + V(\rho^2) \end{aligned} \quad (47)$$

This concludes the gauge independent way of obtaining the final reduced hamiltonian expressed in terms of the independent canonical pairs (ρ, π_ρ) and (Φ, π_Φ) .

Coming next to the issue of gauge fixing, the gauge independent analysis provides the most natural way of proceeding. Remembering that eq. (43) isolated the redundant variable it is clear that the gauge choice,

$$z = 0 \quad (48)$$

would yield a reduced hamiltonian that is trivially equivalent to the gauge independent result (47), with the correspondence $\Phi \rightarrow \phi$; $\pi_\Phi \rightarrow \pi_\phi$. Exactly as happened with the Coulomb gauge in electrodynamics, it transpires that

fixing the $z = 0$ gauge in this model is equivalent to not fixing any gauge.

² It is noteworthy that the authors of [1] have also found this choice (which they have termed as space-axial gauge) to be a more convenient starting point for their analysis than the conventional time-axial gauge $q = 0$. The gauge independent analysis illuminates the reason behind this convenience. Although not considered in [1], there is another equally convenient gauge. This is given by choosing,

$$\phi = 0 \quad (49)$$

It is easy to see from (43) that this choice is on an identical footing to the space axial gauge (48). The canonical variables in this gauge are given by the canonical transformation,

$$\Phi = -gz \quad (50)$$

$$\pi_\Phi = -\frac{1}{g}\pi_z \quad (51)$$

Furthermore it is simple to return to the original coordinates since in this gauge $\rho^2 = x_1^2$ and $\pi_\rho = \pi_1$. The hamiltonian is now written down directly from (47),

$$H = \frac{1}{2}\pi_1^2 + \frac{1}{2}\left(1 + \frac{1}{g^2 x_1^2}\right)\pi_z^2 + V(x_1^2) \quad (52)$$

It is instructive to deduce the above hamiltonian (52) starting from a gauge fixed lagrangian by adopting the standard procedure. This will also show how to deal with more complicated gauges. The gauge (49) is equivalent to taking,

$$x_2 = 0 \quad (53)$$

In that case the solution for the multiplier (40) simplifies to,

$$q = \frac{\dot{z}}{1 + g^2 x_1^2} \quad (54)$$

The reduced Lagrangian after eliminating q from (35) reads,

$$L(x_1, \dot{x}_1) = \frac{1}{2}[\dot{x}_1^2 + \frac{g^2 x_1^2 \dot{z}^2}{1 + g^2 x_1^2}] - V(x_1^2) \quad (55)$$

² In this sense (47) will alternatively be referred to as the hamiltonian in the $z = 0$ gauge.

This Lagrangian is now expressed in terms of the unconstrained variables. The canonical momenta are,

$$\begin{aligned}\pi_1 &= \dot{x}_1 \\ \pi_z &= \frac{g^2 x_1^2 \dot{z}}{1 + g^2 x_1^2}\end{aligned}\tag{56}$$

and the hamiltonian obtained by a Legendre transform is given by,

$$H = \frac{1}{2}\pi_1^2 + \frac{1}{2}\left(1 + \frac{1}{g^2 x_1^2}\right)\pi_z^2 + V(x_1^2)\tag{57}$$

which is just identical to (52). A logical extension of the gauge (53), which follows from the rotational symmetry in the $x_1 - x_2$ plane, would be to consider the choice defined by (20). Since this gauge has been analysed in the context of the CL model we do not pursue it further. It suffices to comment that the reduced hamiltonian is found to be canonically equivalent to the gauge independent result (47), exactly as shown in the case of the CL model.

Let us now try to understand the implications of the above analysis from physical arguments. The constraint (40) can be written as,

$$\pi_z + g\epsilon_{ij}x_i\pi_j = 0\tag{58}$$

where momenta conjugate to the original variables x_i, z in the Lagrangian (35) have been introduced. This implies that the z - component of the angular momentum of the particle is simulated by the z -component of the momentum. It is characteristic of confining the motion to the $x_1 - x_2$ plane. Furthermore, it is clear that this is also equivalent to considering the motion in either the $x_1 - z$ or $x_2 - z$ planes since these just constitute a renaming of the coordinate-axes. Finally, from the rotational symmetry of the problem, it follows that an identical description can be obtained by regarding the motion on any plane normal to the $x_1 - x_2$ plane. All these possibilities have been considered within the gauge independent and subsequent gauge fixed computations. The gauge independent result could be trivially identified with the $z = 0$ gauge, which corresponds to the motion in the $x_1 - x_2$ plane. Likewise the $\phi = 0$ and (20) cases were considered, which correspond to motions in the $x_1 - z$ plane and any plane normal to the $x_1 - x_2$ plane, respectively. All this is

highly reminiscent of our discussion in the CL model. There is, however, an important distinction. In the CL model the gauge (20) fixed the slope of the straight line, which was the trajectory determined from either physical or gauge independent considerations. There was no other possibility. Here, on the other hand, we have not yet exhausted all freedom. It is possible to choose a gauge that forces the motion of the particle to lie on a plane slanted to the $x_1 - x_2$ plane. Such a gauge is defined by,

$$z = \lambda x_1 \quad (59)$$

for any positive λ . It is referred to as the λ gauge [1]. The motion in this gauge cannot be determined easily from physical reasoning. We shall therefore abstract the reduced hamiltonian as done earlier for the $x_2 = 0$ gauge, and then analyse its connection with the gauge independent result (47).

The starting point is to take the Lagrangian (42) obtained after the elimination of the multiplier q . In terms of the polar variables the condition (59) reduces to,

$$z = \lambda \rho \cos \phi \quad (60)$$

from which the time derivative of z may be computed. Inserting this in (42) yields,

$$L(\rho, \dot{\rho}, \phi, \dot{\phi}) = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\beta^{-1}[\alpha^2 \rho^2 \dot{\phi}^2 - 2g\lambda \rho^2 \alpha \dot{\rho} \dot{\phi} \cos \phi + g^2 \lambda^2 \rho^2 \dot{\rho}^2 \cos^2 \phi] \quad (61)$$

where,

$$\begin{aligned} \alpha &= 1 + g\lambda \rho \sin \phi \\ \beta &= 1 + g^2 \rho^2 \end{aligned} \quad (62)$$

The above Lagrangian is now written in terms of the *unconstrained* variables. The canonical momenta are given by,

$$\pi_\rho = \frac{1 + g^2 \rho^2 (1 + \lambda^2 \cos^2 \phi)}{\beta} \dot{\rho} - \frac{g\lambda \rho^2 \alpha \cos \phi}{\beta} \dot{\phi} \quad (63)$$

$$\pi_\phi = \frac{\alpha^2}{\beta} \rho^2 \dot{\phi} - \frac{\alpha}{\beta} g\lambda \rho^2 \dot{\rho} \cos \phi \quad (64)$$

As happens for unconstrained variables, the velocities can be inverted,

$$\dot{\rho} = \pi_{\rho} + \frac{g\lambda \cos \phi}{\alpha} \pi_{\phi} \quad (65)$$

$$\dot{\phi} = \frac{g\lambda \cos \phi}{\alpha} \pi_{\rho} + \frac{g^2 \lambda^2 \rho^2 \cos^2 \phi + \beta}{\rho^2 \alpha^2} \pi_{\phi} \quad (66)$$

Using these expressions the reduced hamiltonian can be found from (61), by taking a suitable Legendre transform,

$$H = \frac{1}{2} [\pi_{\rho}^2 + \frac{2g\lambda \cos \phi}{\alpha} \pi_{\rho} \pi_{\phi} + \frac{g^2 \lambda^2 \rho^2 \cos^2 \phi + \beta}{\rho^2 \alpha^2} \pi_{\phi}^2] + V(\rho^2) \quad (67)$$

This is the final hamiltonian in the λ gauge written in terms of the independent canonical pairs (ρ, π_{ρ}) and (ϕ, π_{ϕ}) . It is simple to verify that if λ is set equal to zero, then the hamiltonian (47) in the $z = 0$ gauge is reproduced. This serves as a consistency check.

We now concentrate on the general form of the hamiltonian for arbitrary λ . As has been stressed the viability of the gauge fixed result must be verified by demonstrating its equivalence, modulo canonical transformations, with the gauge independent analysis. Indeed, in this particular case, it can be checked that the following canonical transformation,

$$\begin{aligned} \pi_{\rho} &= \pi_{\rho} + \frac{g\lambda \cos \phi}{\alpha} \pi_{\phi} \\ \rho &= \rho \\ \pi_{\Phi} &= \alpha^{-1} \pi_{\phi} \\ \Phi &= \phi - g\lambda \rho \cos \phi \end{aligned} \quad (68)$$

maps the gauge independent result (47) with (67). In the above set the canonical pairs on the L.H.S. correspond to (47) while those on the R.H.S. correspond to (67). It is now necessary to check that the above set (68) is nonsingular by working out the corresponding inverse canonical transformation. It is obvious that if the inverse mapping $\phi \rightarrow \Phi$ can be found, the other transformations follow from simple algebraic manipulations. Now in the weak coupling limit $g \rightarrow 0$, this inverse mapping is a trivial identity transformation. Consequently the weak coupling limit of the λ gauge poses no problems and can be regarded on an identical footing as the other gauges.

For arbitrary coupling, however, the inverse mapping $\phi \rightarrow \Phi$ is tricky and has been analysed in great details in ref. [1]. The result is that if,

$$\rho < (\lambda g)^{-1} \quad (69)$$

the mapping is unique. However in those cases when,

$$\rho > (\lambda g)^{-1} \quad (70)$$

the mapping is no longer unique. This implies that although the mapping $\Phi \rightarrow \phi$ is one-to-one, the inverse transformation is one-to-many. Thus the inverse canonical transformation to (68) is nonunique. This is the manifestation of the Gribov ambiguity. Indeed it is precisely for this range (70) of the coupling parameter that the occurrence of the Gribov-like ambiguity was noted in [1]. It is, however, important to point out that the canonical equivalence of the reduced hamiltonian in this gauge with that obtained gauge independently remains valid for all different mappings. In other words this equivalence does not discriminate any specific mapping and treats them all on an identical footing. Our results therefore provide an independent confirmation of the proposal made in [1] to regard all gauge copies equivalently and not to isolate, as suggested by Gribov [2], any special copies.

We now elucidate the connection of our results with the hamiltonian formalism in the Schrödinger representation carried out in [1], based on coordinate transformations from the time axial gauge in which the Cartesian basis is defined. Using these transformations it was shown [1] that the hamiltonian in the space axial gauge $z = 0$ had the form,³

$$H = -\frac{1}{2\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{1}{2} (g^2 + \frac{1}{\rho^2}) \frac{\partial^2}{\partial \Phi^2} + V(\rho^2) \quad (71)$$

It is simple to observe the equivalence of this expression with our result (47) using standard identifications in the Schrödinger representation,

$$\begin{aligned} \pi_\rho^2 &\rightarrow -\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \\ \pi_\Phi &\rightarrow -i \frac{\partial}{\partial \Phi} \end{aligned} \quad (72)$$

³ Henceforth, to avoid notational confusion, we shall refer to the angular variable in the $z = 0$ gauge and the λ gauge by Φ and ϕ , respectively. While the latter symbol has already been used in the discussion from (60) to (67), the former is prompted by the comments made in footnote 2.

where, it may be recalled, the first of these mappings was also used in the analysis of the CL model.

A similar coordinate transformation from the time axial gauge also led to the hamiltonian in the λ gauge. The result was found to be [1],

$$\begin{aligned}
H = & -\frac{\alpha^{-1}}{2\rho} \left[\frac{\partial}{\partial \rho} \rho \alpha \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \rho} \lambda \rho g \cos \phi \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \lambda \rho g \cos \phi \frac{\partial}{\partial \rho} \right. \\
& \left. + \frac{\partial}{\partial \phi} \alpha^{-1} (g^2 \rho + \rho^{-1} + \lambda^2 \rho g^2 \cos^2 \phi) \frac{\partial}{\partial \phi} \right] + V(\rho^2)
\end{aligned} \tag{73}$$

where α has been defined in (62). The above result can be put in the form,

$$\begin{aligned}
H = & -\frac{1}{2\rho} \left[\left(\frac{\partial}{\partial \rho} + g \lambda \cos \phi \alpha^{-1} \frac{\partial}{\partial \phi} \right) \rho \left(\frac{\partial}{\partial \rho} + g \lambda \cos \phi \alpha^{-1} \frac{\partial}{\partial \phi} \right) \right] \\
& - \frac{1}{2} \left(g^2 + \frac{1}{\rho^2} \right) \alpha^{-2} \left(\frac{\partial^2}{\partial \phi^2} - g \rho \lambda \cos \phi \alpha^{-1} \frac{\partial}{\partial \phi} \right) + V(\rho^2)
\end{aligned} \tag{74}$$

It is now easy to realise that this expression follows directly from the hamiltonian (71) in the $z = 0$ gauge by making the change of variables,

$$\begin{aligned}
\frac{\partial}{\partial \Phi} &= \alpha^{-1} \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \rho} &= \frac{\partial}{\partial \rho} + g \lambda \cos \phi \alpha^{-1} \frac{\partial}{\partial \phi}
\end{aligned} \tag{75}$$

These relations are the analogues of our canonical transformations (68) which map the hamiltonian computed in a gauge independent manner with the hamiltonian in the λ gauge ($z = \lambda x_1$). Indeed, exploiting this analogy, it is possible to define the operator ordering in (67) so that the quantum theory obtained from it is in one-to-one correspondence with the Schrodinger representation result, choosing the Cartesian basis in the time axial gauge. It is clear that the issue of operator ordering is relevant only for the hamiltonian (67). The gauge independent result (47), for instance, is free of any ordering ambiguities. Since the gauge fixed result could be obtained from the gauge independent one using canonical transformations (68), it is evident that the origin of the ordering problem is contained in these transformations. A simple inspection shows that this is indeed true. By comparing (75) with (68), and recalling the identification (72), it is found that the operators in (68)

should be ordered such that the canonical momenta occur after the coordinate variables. In fact the transformations (68) are already written with this prescription. If we now reconstruct the hamiltonian in the λ gauge from the gauge independent expression (47) keeping the ordering in (68) intact, we obtain,

$$\begin{aligned}
H &= \frac{1}{2}[\pi_\rho^2 + \frac{g\lambda \cos \phi}{\alpha} \pi_\phi \pi_\rho + \pi_\rho \frac{g\lambda \cos \phi}{\alpha} \pi_\phi + \\
&+ \frac{g\lambda \cos \phi}{\alpha} \pi_\phi \frac{g\lambda \cos \phi}{\alpha} \pi_\phi + \frac{\beta}{\rho^2} \frac{1}{\alpha} \pi_\phi \frac{1}{\alpha} \pi_\phi] + V(\rho^2)
\end{aligned} \tag{76}$$

It is simple to see that the terms quadratic in either π_ρ or π_ϕ in the above hamiltonian are mapped to their corresponding structures in (74) on using the Schrödinger representations (72). The cross terms involving the product of π_ρ and π_ϕ may now be identified by inspection. This shows that (76) is the quantum hamiltonian corresponding to the classical expression (67) such that compatibility with the Schrödinger representation form (74) is preserved.

IV. The Dirac Analysis

It is well known that, besides the hamiltonian formalism utilising the Schrödinger representation, there is an alternative hamiltonian approach which is based on Dirac's [9] analysis of constrained systems. We discuss Friedberg et al's model in this context and show how the complications in the λ - gauge arise. Since the time derivative of q does not appear in (35), the momentum conjugate to it is a constraint,

$$\pi_q = 0 \tag{77}$$

In Dirac's nomenclature [9], this is a primary constraint. There is a secondary constraint which is found by time-conserving (77) with the hamiltonian,

$$\begin{aligned}
H &= \pi_i \dot{x}_i + \pi_z \dot{z} + \pi_q \dot{q} - L \\
&= \frac{1}{2}(\pi_i^2 + \pi_z^2) + q(\pi_z + g\epsilon_{ij}x_i\pi_j) + V(\rho^2)
\end{aligned} \tag{78}$$

obtained by taking a Legendre transform of (35). This secondary constraint, enforced by the Lagrange multiplier q , has already appeared in (58). There are no further constraints. These constraints are first class since their Poisson brackets are involutive. Indeed, as recognised earlier, the secondary constraint is just the generator of the gauge transformations (14) and (39). Corresponding to the two constraints there are two gauge fixing conditions. One of these conditions must involve q so that it has a nonvanishing bracket with (77). A simple choice is provided by,

$$q = 0 \quad (79)$$

We shall next modify the other gauge condition to first consider the $z = 0$ gauge and then the λ gauge ($z = \lambda x_1$). In the first case the full set of constraints is now given by (58), (77), (79) and $z = 0$. Here the canonical pairs are easily identifiable. A straightforward computation of the Dirac brackets ⁴ (denoted by a star) reveals that these are given by (x_1, π_1) and (x_2, π_2) because,

$$\begin{aligned} \{x_i, \pi_j\}^* &= \delta_{ij} \\ \{x_i, x_j\}^* &= \{\pi_i, \pi_j\}^* = 0 \end{aligned} \quad (80)$$

The brackets involving the other variables are redundant since these drop out from the physical hamiltonian (H_p) which corresponds to the evaluation of (78) on the constraint shell (58). It is given by,

$$H_p = \frac{1}{2}(\pi_1^2(1 + g^2 x_2^2) + \pi_2^2(1 + g^2 x_1^2) - 2g^2 x_1 x_2 \pi_1 \pi_2) + V(\rho^2) \quad (81)$$

It is straightforward to show the equivalence of this hamiltonian, modulo canonical transformations, with the expression (47). These transformations are given by,

$$\begin{aligned} x_1 &= \rho \cos \Phi \\ x_2 &= \rho \sin \Phi \\ \pi_1 &= \pi_\rho \cos \Phi - \frac{\pi_\Phi}{\rho} \sin \Phi \\ \pi_2 &= \pi_\rho \sin \Phi + \frac{\pi_\Phi}{\rho} \cos \Phi \end{aligned} \quad (82)$$

⁴These brackets are defined by the standard formula (88)

There is no ambiguity in obtaining the inverse canonical transformations which are given by,

$$\begin{aligned}
\rho^2 &= x_1^2 + x_2^2 \\
\Phi &= \tan^{-1} \frac{x_2}{x_1} \\
\pi_\rho &= \frac{x_1 \pi_1 + x_2 \pi_2}{\sqrt{x_1^2 + x_2^2}} \\
\pi_\Phi &= \epsilon_{ij} x_i \pi_j
\end{aligned} \tag{83}$$

Interestingly, such transformations were mentioned earlier [6] in the hamiltonian reduction of the CL model that was based on the Levi-Civita approach.

We now consider the λ gauge. The analysis in the original Cartesian variables is quite cumbersome and not very illuminating. Taking a hint from the previous analysis, however, the problem can be simplified. A change of variables in the original Lagrangian (35) is made by using the standard polar decomposition given by the the first pair of equations in (82). It is now possible to compute the constraint (58) in these variables by working anew the Dirac prescription. But a short cut can be taken by realising that the corresponding momenta are just the last pair of equations in (83). The constraint is therefore given by,

$$\pi_z + g\epsilon_{ij}x_i\pi_j = \pi_z + g\pi_\phi = 0 \tag{84}$$

The complete set of constraints $\Omega_i = 0$ in the λ gauge is now explicitly enumerated,

$$\begin{aligned}
\Omega_1 &= \pi_q \\
\Omega_2 &= q \\
\Omega_3 &= z - \lambda\rho \cos \phi \\
\Omega_4 &= \pi_z + g\pi_\phi
\end{aligned} \tag{85}$$

Since the Dirac brackets are nontrivial it is useful to give some algebraic details. The matrix of the Poisson brackets of Ω_i is given by,

$$\begin{aligned}
\Omega_{ij} &= \{\Omega_i, \Omega_j\} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}
\end{aligned} \tag{86}$$

where α has been defined in (62). The inverse matrix is easily computed,

$$\begin{aligned}\Omega_{ij}^{-1} &= (\{\Omega_i, \Omega_j\})^{-1} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha^{-1} \\ 0 & 0 & \alpha^{-1} & 0 \end{pmatrix}\end{aligned}\quad (87)$$

Dirac brackets among any two variables A and B can be calculated using the formula [9],

$$\{A, B\}^* = \{A, B\} - \{A, \Omega_i\} \Omega_{ij}^{-1} \{\Omega_j, B\} \quad (88)$$

The nontrivial Dirac brackets (i.e. those differing from the corresponding Poisson brackets) are explicitly written,

$$\begin{aligned}\{\phi, \pi_\phi\}^* &= \alpha^{-1} \\ \{\phi, \pi_\rho\}^* &= (g\lambda \cos \phi) \alpha^{-1}\end{aligned}\quad (89)$$

As a simple consistency check of this algebra, it is straightforward to reproduce the results in the $z = 0$ gauge by setting $\lambda = 0$. In this case $\alpha = 1$ and the canonical pairs are (Φ, π_Φ) and (ρ, π_ρ) . This was realised earlier in the canonical transformations (82) mapping (81) to (47).

To proceed for arbitrary λ , it is first essential to identify the canonical set. This is done by exploiting the algebra,

$$\begin{aligned}\{\phi - g\lambda\rho \cos \phi, \pi_\phi\}^* &= 1 \\ \{\phi - g\lambda\rho \cos \phi, \pi_\rho\}^* &= 0\end{aligned}\quad (90)$$

deduced from the basic brackets (89). Consequently the independent canonical pairs in the λ gauge are (ρ, π_ρ) and $(\phi - g\lambda\rho \cos \phi, \pi_\phi)$. With this knowledge it is feasible to reconstruct the hamiltonian in the λ gauge from the result in the $z = 0$ gauge. A comparison of the canonical sets in these two gauges immediately leads to the following mappings in the Schrodinger representation,

$$\begin{aligned}\Phi &= \phi - g\lambda\rho \cos \phi \\ \frac{\partial}{\partial \Phi} &= \alpha^{-1} \frac{\partial}{\partial \phi} \\ \rho &= \rho \\ \frac{\partial}{\partial \rho} &= \frac{\partial}{\partial \rho} + g\lambda \cos \phi \alpha^{-1} \frac{\partial}{\partial \phi}\end{aligned}\quad (91)$$

such that compatibility with (90) is preserved. The hamiltonian in the $z = 0$ gauge was already shown to be canonically equivalent to (47) which, in the Schrödinger representation, is given by (71). With the above mapping (91), the hamiltonian (74) in the λ gauge is reproduced from (71). This becomes obvious if it is realised that (91) is identical to the mapping (75), found on the basis of coordinate transformations, which was shown earlier to connect the hamiltonian in the two gauges. Consequently the correspondence between the two hamiltonian approaches, based on either Dirac's [9] analysis or using coordinate transformations in the Schrödinger representation [1], is established.

V. Conclusions

Using the gauge independent method, recently developed by us [8], of reducing the degrees of freedom in a gauge theory we have presented a detailed analysis of a solvable model with Gribov-like ambiguity, proposed by Friedberg, Lee, Pang and Ren [1] (called here as 'Extended Christ Lee Model'). This reduction was carried out at the Lagrangian level by systematically eliminating the redundant or unphysical degrees of freedom in a two-step process. First, the multiplier that enforces the Gauss constraint was eliminated by solving this constraint. Next, a change of variables was effected which eliminated the cyclic coordinate that was responsible for the degeneracy in the equations of motion that is characteristic of any gauge theory. This change of variables was dictated by the nature of the gauge symmetry. For example in QED where this symmetry is translational, a simple shift (8) was required, whereas in either Friedberg et al's model [1] or the Christ Lee (CL) model [10] where the symmetry is rotational, a curvilinear transformation was necessary. In all instances the cyclic coordinate was identified and naturally eliminated from the Lagrangian which was now expressed solely in terms of unconstrained variables. The physical hamiltonian was computed directly from this unconstrained Lagrangian.

The reduced space was also obtained by fixing a gauge whose effect was to impose certain restrictions on the unphysical variables. The viability of the gauge fixed computations was judged by checking the canonical equivalence

with the results obtained gauge independently. In fact the relevant canonical transformations were explicitly worked out for both the CL model and Friedberg et al's model. Interestingly, in the latter case it was found that the canonical transformations in the λ gauge ($z = \lambda x_1$) did not possess a unique inverse, thereby yielding a Gribov-like phenomenon. Our analysis showed that all gauge copies must be treated equivalently, which corroborated earlier findings [1], [3]. The connection of the canonical transformations discussed here with the coordinate transformations in the usual hamiltonian formalism employing the Schrödinger representation [4] was clearly revealed for either Gribov ambiguity free gauges or the λ gauge. Using this connection the operator ordering problem in the λ gauge was resolved.

As an alternative hamiltonian formalism, Friedberg et al's model was also investigated using Dirac's [9] constrained analysis. The occurrence of a nontrivial canonical set in the λ gauge, as opposed to the set in the $z = 0$ gauge, led to a mapping relating the physical hamiltonians in these gauges. In the Schrödinger representation this mapping was exactly identical to that found by using Friedberg et al's analysis.

Coming back to the gauge independent analysis it provided a natural way of understanding those reduction process in these models that were dictated from purely physical arguments. Furthermore, a simple way of identifying a convenient gauge to be employed in gauge fixed computations was an important consequence of the gauge independent analysis. Just as the Coulomb gauge was shown to be the natural choice in QED, the $z = 0$ gauge was the corresponding choice in Friedberg et al's model. Significantly, the authors of [1] have found this choice, instead of the conventional time-axial gauge ($q = 0$), as a more convenient starting point for their hamiltonian formulation. In more complicated theories, therefore, the guideline provided by the gauge independent analysis concerning gauge fixing could be valuable.

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